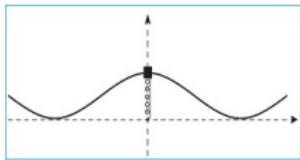


# How is an EFT for Gravitational Waves in any dimension like a bead on a string?



Ofek Birnholtz  
with S. Hadar and B. Kol



- O.B., S.H. & B.K., Phys. Rev. D 88, 104037 (2013)  
O.B. & S.H., Phys. Rev. D 89, 045003 (2014)  
O.B., S.H. & B.K., IJMPA 29, 24, 30 (2014)

Bi-group meeting, Humboldt University, Berlin

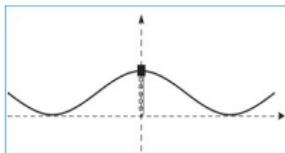
20 October 2014

## Outline

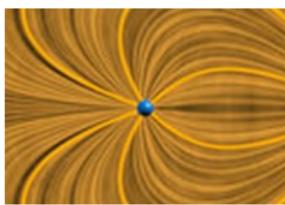
- 1 Motivation
- 2 A bead on a string
- 3 Electromagnetism: the ALD self-force in any dimension
- 4 Gravitational Waves in d dimensions
- 5 Conclusions

“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein

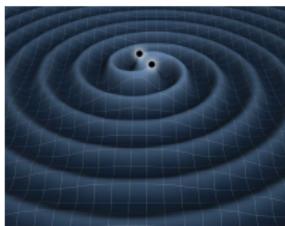
“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein



Bead + string (elastic field)



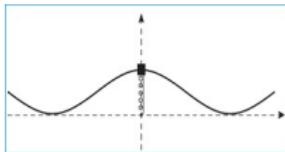
Electric charge + EM field



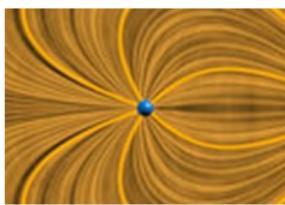
2-Body system + GR field

“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein

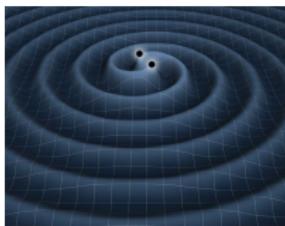
$$\hat{S} = \left\| \text{curly line} \right\| = \sum (\hat{Q} G Q)$$



Bead + string (elastic field)



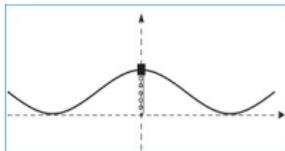
Electric charge + EM field



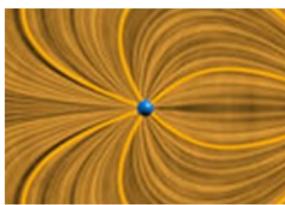
2-Body system + GR field

“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein

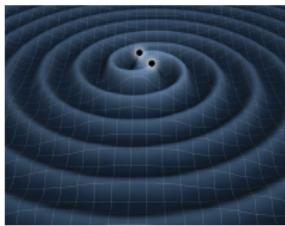
$$\hat{S} = \left\| \text{---} \right\| = \sum (\hat{Q} G Q) , \quad F = \frac{\delta \hat{S}}{\delta \hat{x}}$$



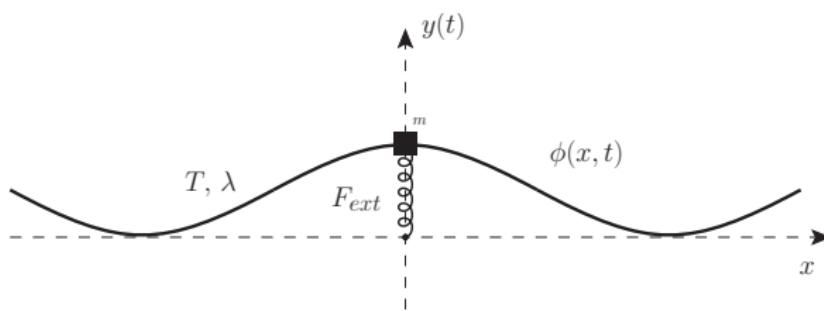
Bead + string (elastic field)



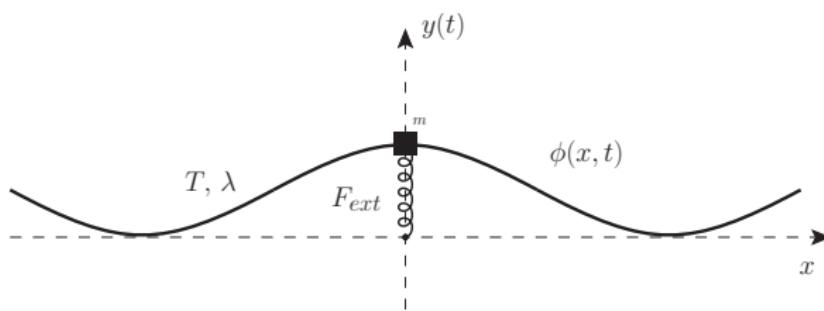
Electric charge + EM field



2-Body system + GR field

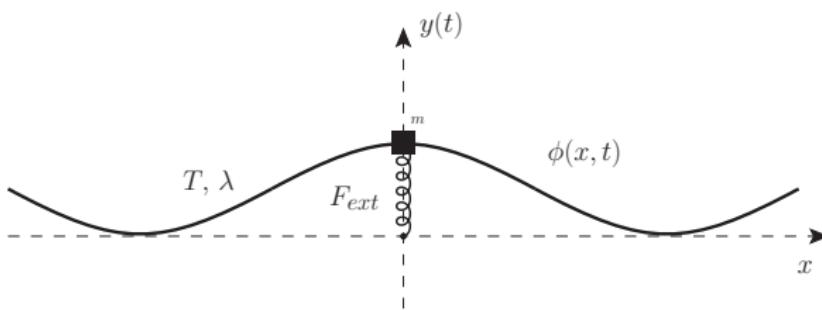


$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$



$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

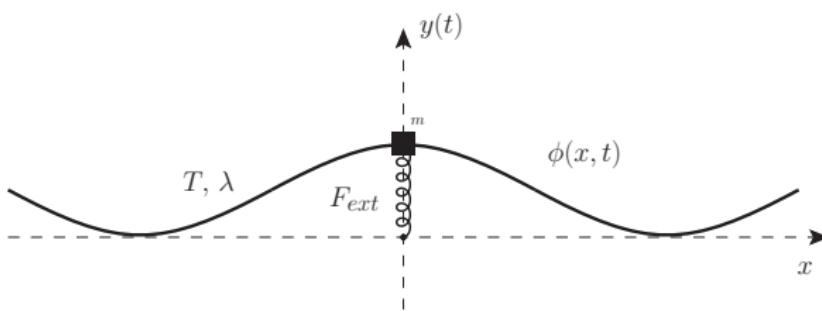
$$\implies \phi(x=0, t) = y(t),$$



$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

$$\implies \phi(x=0, t) = y(t),$$

$$\square \phi = [c^{-2} \partial_t^2 - \partial_x^2] \phi = -\delta(x) Q(t)/T \quad , \quad c^2 = T/\lambda$$

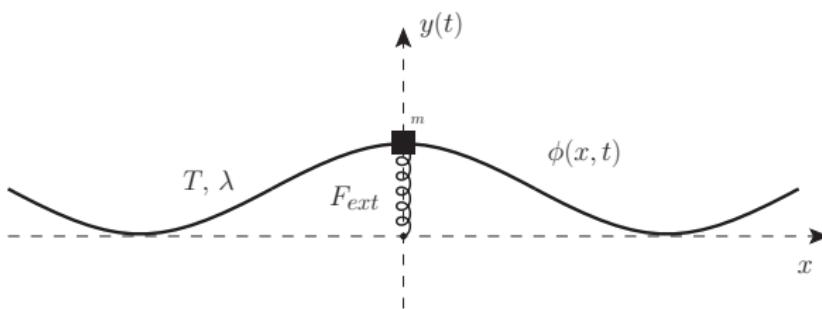


$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

$$\implies \phi(x=0, t) = y(t),$$

$$\square \phi = [c^{-2} \partial_t^2 - \partial_x^2] \phi = -\delta(x) Q(t)/T \quad , \quad c^2 = T/\lambda$$

$$\implies \phi(x, t) = \begin{cases} y(t - x/c) & x > 0 \\ y(t + x/c) & x < 0 \end{cases}$$



$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

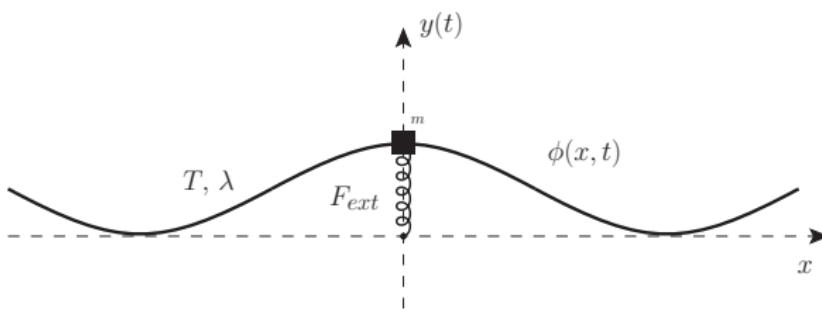
$$\Rightarrow \phi(x=0, t) = y(t),$$

$$\square \phi = [c^{-2} \partial_t^2 - \partial_x^2] \phi = -\delta(x) Q(t)/T , \quad c^2 = T/\lambda$$

$$\Rightarrow \phi(x, t) = \begin{cases} y(t - x/c) & x > 0 \\ y(t + x/c) & x < 0 \end{cases}$$

$$\Rightarrow Q = T [(\partial_x \phi)|_{0+} - (\partial_x \phi)|_{0-}] = -2Z \dot{y}, \quad Z = \sqrt{\lambda T}$$

$$\Rightarrow m \ddot{y} = -U'_{\text{ext}} + Q = F_{\text{ext}} - 2Z \dot{y}$$



$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

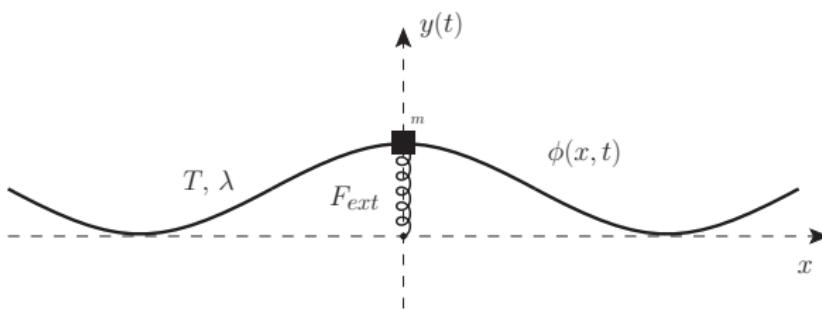
$$\Rightarrow \phi(x=0, t) = y(t),$$

$$\square \phi = [c^{-2} \partial_t^2 - \partial_x^2] \phi = -\delta(x) Q(t)/T , \quad c^2 = T/\lambda$$

$$\Rightarrow \phi(x, t) = \begin{cases} y(t - x/c) & x > 0 \\ y(t + x/c) & x < 0 \end{cases}$$

$$\Rightarrow Q = T [(\partial_x \phi)|_{0+} - (\partial_x \phi)|_{0-}] = -2Z \dot{y} , \quad Z = \sqrt{\lambda T}$$

$$\Rightarrow m \ddot{y} = -U'_{\text{ext}} + Q = F_{\text{ext}} - 2Z \dot{y}$$



$$S = \int dt \left[ \frac{m}{2} \dot{y}^2 - U_{\text{ext}} \right] + \int dt dx \left[ \frac{\lambda}{2} \dot{\phi}^2 - \frac{T}{2} \phi'^2 \right] - \int dt Q(t) [\phi(x=0) - y]$$

$$\Rightarrow \phi(x=0, t) = y(t),$$

$$\square \phi = [c^{-2} \partial_t^2 - \partial_x^2] \phi = -\delta(x) Q(t)/T , \quad c^2 = T/\lambda$$

$$\Rightarrow \phi(x, t) = \begin{cases} y(t - x/c) & x > 0 \\ y(t + x/c) & x < 0 \end{cases}$$

$$\Rightarrow Q = T [(\partial_x \phi)|_{0+} - (\partial_x \phi)|_{0-}] = -2Z \dot{y}, \quad Z = \sqrt{\lambda T}$$

$$\Rightarrow m \ddot{y} = -U'_{\text{ext}} + Q = F_{\text{ext}} - 2Z \dot{y}$$

## Field doubling at the level of the Action

An effective action for  $\phi$  (kinetic term & source term):

$$S = \frac{T}{2} \int dt dx \left[ \frac{1}{c^2} \dot{\phi}^2 - \phi'^2 \right] + 2 Z \int dt \dot{y} \phi(x=0)$$

## Field doubling at the level of the Action

An effective action for  $\phi$  (kinetic term & source term):

$$S = \frac{T}{2} \int dt dx \left[ \frac{1}{c^2} \dot{\phi}^2 - \phi'^2 \right] + 2 Z \int dt \dot{y} \phi(x=0)$$

Double the field and the source (Schwinger-Keldysh, Galley) :

- These reflect directed propagation
- Interpretation: "pulling" mirror / radiation "sink"

$$\phi \rightarrow \hat{\phi} \quad , \quad Q \rightarrow \hat{Q} = \frac{\delta Q}{\delta y} \hat{y}$$



## Field doubling at the level of the Action

An effective action for  $\phi$  (kinetic term & source term):

$$S = \frac{T}{2} \int dt dx \left[ \frac{1}{c^2} \dot{\phi}^2 - \phi'^2 \right] + 2 Z \int dt \dot{y} \phi(x=0)$$

Double the field and the source (Schwinger-Keldysh, Galley) :

- These reflect directed propagation
- Interpretation: "pulling" mirror / radiation "sink"

$$\phi \rightarrow \hat{\phi} \quad , \quad Q \rightarrow \hat{Q} = \frac{\delta Q}{\delta y} \hat{y}$$



Doubled action:

$$\begin{aligned} \hat{S} [\phi, \hat{\phi}; Q, \hat{Q}] &= \int d^d x \left[ \frac{\delta S}{\delta \phi} \hat{\phi} + \frac{\delta S}{\delta Q} \hat{Q} \right] \\ &= \int \frac{d\omega}{2\pi} \left[ \frac{Z}{c} \int dx \hat{\phi} (\omega^2 + \partial_x^2) \phi - 2i\omega Z \left( y\hat{\phi}(0) + \hat{y}\phi(0) \right) \right] \end{aligned}$$

EOM found by varying w.r.t.  $\hat{\phi}$

## Feynman rules

Directed propagator (non-hatted  $\rightarrow$  hatted)

$$\text{Diagram: A wavy line from } x' \text{ to } x \text{ with a label } L. \quad = G_\omega(x', x) = -c \frac{e^{i\frac{\omega}{c}|x-x'|}}{i\omega Z}$$

Source vertex

$$\text{Diagram: Two parallel vertical lines with a wavy line connecting them.} \quad -Q_\omega \equiv \parallel \text{---} \text{Wavy Line} \parallel = -i\omega Z y_\omega$$

Hatted source vertex

$$\text{Diagram: Two parallel vertical lines with a wavy line connecting them.} \quad -\hat{Q}_\omega^* \equiv \text{Wavy Line} \parallel = +i\omega Z \hat{y}_\omega^*$$

## A bead on a string - results

Radiation ( $x > 0$ )

$$\phi_\omega(x) = \left| \int \right|^x = c y_\omega e^{i\omega x/c} \implies \phi(t, x) = y\left(t - \frac{x}{c}\right)$$

## A bead on a string - results

Radiation ( $x > 0$ )

$$\phi_\omega(x) = \left\| \int \right\|^\infty_x = c y_\omega e^{i\omega x/c} \implies \phi(t, x) = y(t - \frac{x}{c})$$

Radiation-Reaction effective action

$$\begin{aligned}\hat{S}_{RR} &= \left\| \int \right\| + \text{c.c.} = \int \frac{d\omega}{2\pi} \hat{Q}_\omega^* G_\omega(0, 0) Q_\omega + \text{c.c.} \\ &= \int \frac{d\omega}{2\pi} i\omega Z \hat{y}_\omega^* y_\omega + \text{c.c.} = -2Z \int \hat{y} \dot{y} dt\end{aligned}$$

## A bead on a string - results

Radiation ( $x > 0$ )

$$\phi_\omega(x) = \left\| \int \right\|^\infty_x = c y_\omega e^{i\omega x/c} \implies \phi(t, x) = y(t - \frac{x}{c})$$

Radiation-Reaction effective action

$$\begin{aligned}\hat{S}_{RR} &= \left\| \int \right\| + \text{c.c.} = \int \frac{d\omega}{2\pi} \hat{Q}_\omega^* G_\omega(0, 0) Q_\omega + \text{c.c.} \\ &= \int \frac{d\omega}{2\pi} i\omega Z \hat{y}_\omega^* y_\omega + \text{c.c.} = -2Z \int \hat{y} \dot{y} dt\end{aligned}$$

Radiation-Reaction (Self-) Force

$$F_{RR} = \frac{\delta \hat{S}_{RR}}{\delta \hat{y}} = -2Z \dot{y}$$

## A bead on a string - results

Radiation ( $x > 0$ )

$$\phi_\omega(x) = \left\| \int \right\|^\infty_x = c y_\omega e^{i\omega x/c} \implies \phi(t, x) = y(t - \frac{x}{c})$$

Radiation-Reaction effective action

$$\begin{aligned} \hat{S}_{RR} &= \left\| \int \right\| + \text{c.c.} = \int \frac{d\omega}{2\pi} \hat{Q}_\omega^* G_\omega(0, 0) Q_\omega + \text{c.c.} \\ &= \int \frac{d\omega}{2\pi} i\omega Z \hat{y}_\omega^* y_\omega + \text{c.c.} = -2Z \int \hat{y} \dot{y} dt \end{aligned}$$

Radiation-Reaction (Self-) Force

$$F_{RR} = \frac{\delta \hat{S}_{RR}}{\delta \hat{y}} = -2Z \dot{y}$$

## A bead on a string - results

Radiation ( $x > 0$ )

$$\phi_\omega(x) = \left\| \int \right\|^\infty_x = c y_\omega e^{i\omega x/c} \implies \phi(t, x) = y(t - \frac{x}{c})$$

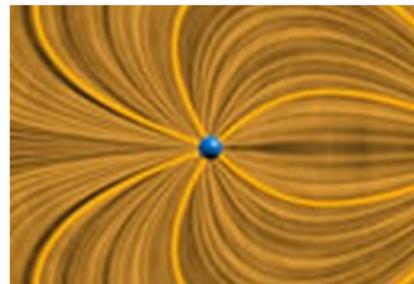
Radiation-Reaction effective action

$$\begin{aligned}\hat{S}_{RR} &= \left\| \int \right\| + \text{c.c.} = \int \frac{d\omega}{2\pi} \hat{Q}_\omega^* G_\omega(0, 0) Q_\omega + \text{c.c.} \\ &= \int \frac{d\omega}{2\pi} i\omega Z \hat{y}_\omega^* y_\omega + \text{c.c.} = -2Z \int \hat{y} \dot{y} dt\end{aligned}$$

Radiation-Reaction (Self-) Force

$$F_{RR} = \frac{\delta \hat{S}_{RR}}{\delta \hat{y}} = -2Z \dot{y} \quad \text{damping}$$

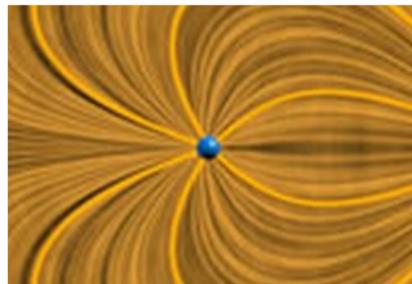
## Electromagnetism in d dimensions



$$\begin{aligned} S = & -\frac{1}{4\Omega_{\hat{d}+1}} \int F_{\mu\nu} F^{\mu\nu} d^d x \\ & - \int A_\mu J^\mu d^d x \end{aligned}$$

( $D := d - 1$  ,  $\hat{d} := d - 3$  ,  $c = 1$ )

## Electromagnetism in d dimensions

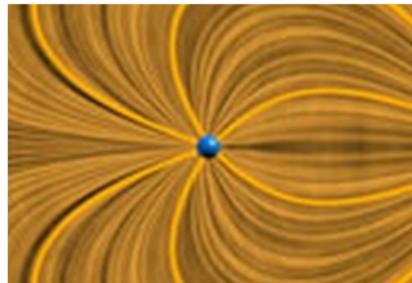


$$\begin{aligned} S = & -\frac{1}{4\Omega_{\hat{d}+1}} \int F_{\mu\nu} F^{\mu\nu} d^d x \\ & - \int A_\mu J^\mu d^d x \end{aligned}$$

( $D := d - 1$  ,  $\hat{d} := d - 3$  ,  $c = 1$ )

Different, more, or both?

## Electromagnetism in d dimensions



$$\begin{aligned} S = & -\frac{1}{4\Omega_{\hat{d}+1}} \int F_{\mu\nu} F^{\mu\nu} d^d x \\ & - \int A_\mu J^\mu d^d x \end{aligned}$$

(D := d - 1 ,  $\hat{d} := d - 3$  , c = 1)

Different, more, or both?

In the right basis, only a little more, and a little different!

## Electromagnetism - intermediate basis (have no fear!)

Multipoles = (solid) spherical harmonics, scalar & vector

$$A_{t/r} = \int \frac{d\omega}{2\pi} \sum_L A_{t/r}^{L\omega} x_L e^{-i\omega t}, \quad A_\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( A_S^{L\omega} \partial_\Omega x_L + A_{VN}^{L\omega} x_N^\Omega \right) e^{-i\omega t}$$

$$J^{t/r} = \int \frac{d\omega}{2\pi} \sum_L J_{L\omega}^{t/r} x^L e^{-i\omega t}, \quad J^\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( J_S^{L\omega} \partial^\Omega x^L + J_{L\omega}^{VN} x_L^\Omega \right) e^{-i\omega t}$$

$$L = (k_1 k_2 \cdots k_\ell)^{\text{STF}} \quad , \quad x^L = \left( x^{k_1} x^{k_2} \cdots x^{k_\ell} \right)^{\text{STF}} \sim r^\ell Y_{\ell\vec{m}}(\Omega)$$

## Electromagnetism - intermediate basis (have no fear!)

Multipoles = (solid) spherical harmonics, scalar & vector

$$A_{t/r} = \int \frac{d\omega}{2\pi} \sum_L A_{t/r}^{L\omega} x_L e^{-i\omega t}, \quad A_\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( A_S^{L\omega} \partial_\Omega x_L + A_{VN}^{L\omega} x_N^\Omega \right) e^{-i\omega t}$$

$$J^{t/r} = \int \frac{d\omega}{2\pi} \sum_L J_{L\omega}^{t/r} x^L e^{-i\omega t}, \quad J^\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( J_S^{L\omega} \partial^\Omega x^L + J_{L\omega}^{VN} x_L^\Omega \right) e^{-i\omega t}$$

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L N_{\ell,\hat{d}} S_{L\omega}$$

$$\begin{aligned} S_{L\omega} = & \int dr r^{2\ell+\hat{d}+1} \left\{ \left[ \left| i\omega A_r^{L\omega} - \frac{1}{r^\ell} (r^\ell A_t^{L\omega})' \right|^2 + \frac{c_s}{r^2} \left| i\omega A_S^{L\omega} - A_t^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_S^{L\omega})' - A_r^{L\omega} \right|^2 \right. \right. \\ & + c_s \left( \frac{\omega^2}{r^2} - \frac{c_v}{r^4} \right) \left| A_{VN}^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_{VN}^{L\omega})' \right|^2 \left. \right] \\ & - \Omega_{\hat{d}+1} \left[ A_r^{L\omega} J_{L\omega}^{r*} + A_t^{L\omega} J_{L\omega}^{t*} + c_s A_S^{L\omega} J_{L\omega}^{s*} + c_s A_{VN}^{L\omega} J_{L\omega}^{VN*} + c.c. \right] \end{aligned}$$

$$N_{\ell,\hat{d}} = \frac{\Gamma(1+\hat{d}/2)}{2^\ell \Gamma(\ell+1+\hat{d}/2)} = \frac{\hat{d}!!}{(2\ell+\hat{d})!!}, \quad c_s := \ell(\ell+\hat{d}), \quad c_v = c_s + \hat{d} - 1$$

## Electromagnetism - intermediate basis (have no fear!)

Multipoles = (solid) spherical harmonics, scalar & vector

$$A_{t/r} = \int \frac{d\omega}{2\pi} \sum_L A_{t/r}^{L\omega} x_L e^{-i\omega t}, \quad A_\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( A_S^{L\omega} \partial_\Omega x_L + A_{VN}^{L\omega} x_N^\Omega \right) e^{-i\omega t}$$

$$J^{t/r} = \int \frac{d\omega}{2\pi} \sum_L J_{L\omega}^{t/r} x^L e^{-i\omega t}, \quad J^\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( J_S^{L\omega} \partial^\Omega x^L + J_{L\omega}^{VN} x_L^\Omega \right) e^{-i\omega t}$$

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L N_{\ell,\hat{d}} S_{L\omega}$$

$$\begin{aligned} S_{L\omega} = & \int dr r^{2\ell+\hat{d}+1} \left\{ \left[ \left| i\omega A_r^{L\omega} - \frac{1}{r^\ell} (r^\ell A_t^{L\omega})' \right|^2 + \frac{c_s}{r^2} \left| i\omega A_S^{L\omega} - A_t^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_S^{L\omega})' - A_r^{L\omega} \right|^2 \right. \right. \\ & + c_s \left( \frac{\omega^2}{r^2} - \frac{c_v}{r^4} \right) \left| A_{VN}^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_{VN}^{L\omega})' \right|^2 \left. \right] \\ & - \Omega_{\hat{d}+1} \left[ A_r^{L\omega} J_{L\omega}^{r*} + A_t^{L\omega} J_{L\omega}^{t*} + c_s A_S^{L\omega} J_{L\omega}^{S*} + c_s A_{VN}^{L\omega} J_{L\omega}^{VN*} + c.c. \right] \end{aligned}$$

$$N_{\ell,\hat{d}} = \frac{\Gamma(1+\hat{d}/2)}{2^\ell \Gamma(\ell+1+\hat{d}/2)} = \frac{\hat{d}!!}{(2\ell+\hat{d})!!}, \quad c_s := \ell(\ell+\hat{d}), \quad c_v = c_s + \hat{d} - 1$$

## Electromagnetism - intermediate basis (have no fear!)

Multipoles = (solid) spherical harmonics, scalar & vector

$$A_{t/r} = \int \frac{d\omega}{2\pi} \sum_L A_{t/r}^{L\omega} x_L e^{-i\omega t}, \quad A_\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( A_S^{L\omega} \partial_\Omega x_L + A_{VN}^{L\omega} x_N^\Omega \right) e^{-i\omega t}$$

$$J^{t/r} = \int \frac{d\omega}{2\pi} \sum_L J_{L\omega}^{t/r} x^L e^{-i\omega t}, \quad J^\Omega = \int \frac{d\omega}{2\pi} \sum_L \left( J_S^{L\omega} \partial^\Omega x^L + J_{L\omega}^{VN} x_L^\Omega \right) e^{-i\omega t}$$

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L N_{\ell,\hat{d}} S_{L\omega}$$

$$\begin{aligned} S_{L\omega} = & \int dr r^{2\ell+\hat{d}+1} \left\{ \left[ \left| i\omega A_r^{L\omega} - \frac{1}{r^\ell} (r^\ell A_t^{L\omega})' \right|^2 + \frac{c_s}{r^2} \left| i\omega A_S^{L\omega} - A_t^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_S^{L\omega})' - A_r^{L\omega} \right|^2 \right. \right. \\ & + c_s \left( \frac{\omega^2}{r^2} - \frac{c_v}{r^4} \right) \left| A_{VN}^{L\omega} \right|^2 - \frac{c_s}{r^2} \left| \frac{1}{r^\ell} (r^\ell A_{VN}^{L\omega})' \right|^2 \left. \right] \\ & - \Omega_{\hat{d}+1} \left[ A_r^{L\omega} J_{L\omega}^{r*} + A_t^{L\omega} J_{L\omega}^{t*} + c_s A_S^{L\omega} J_{L\omega}^{S*} + c_s A_{VN}^{L\omega} J_{L\omega}^{VN*} + c.c. \right] \end{aligned}$$

$$N_{\ell,\hat{d}} = \frac{\Gamma(1+\hat{d}/2)}{2^\ell \Gamma(\ell+1+\hat{d}/2)} = \frac{\hat{d}!!}{(2\ell+\hat{d})!!}, \quad c_s := \ell(\ell+\hat{d}), \quad c_v = c_s + \hat{d} - 1$$

## Electromagnetism - reduction to gauge invariant fields 1

Scalar field & source:

$$A_{L\omega}^r = -\frac{1}{\omega^2 - \frac{c_s}{r^2}} \left[ \frac{i\omega}{r^\ell} (r^\ell A_{L\omega}^t)' + \frac{c_s}{r^{\ell+2}} (r^\ell A_{L\omega}^S)' - \Omega_{\hat{d}+1} J_{L\omega}^r \right]$$

$$\tilde{A}_S^{L\omega} := A_t^{L\omega} - i\omega A_S^{L\omega}$$

$$\rho_{L\omega}^S := -J_{L\omega}^t + \frac{i}{\omega r^{\ell+\hat{d}+1}} \left( r^{\ell+\hat{d}+1} \frac{\Lambda}{\Lambda-1} J_{L\omega}^r \right)', \quad \Lambda := \frac{\omega^2 r^2}{c_s}$$

Vector field & source:

$$A_{VN}^{L\omega}, \quad \rho_{L\omega}^{VN} := J_{L\omega}^{VN}$$

## Electromagnetism - reduction to gauge invariant fields 2

- Scalar (“Electric”) field

$$\begin{aligned}
 A_E^{L\omega} &= \left( \ell r^{\ell-1} (1 - \Lambda) \right)^{-1} \left( r^\ell (A_t^{L\omega} - i\omega A_S^{L\omega}) \right)' \\
 \rho_{L\omega}^{A_E} &= \int d\Omega_{\hat{d}+1} \frac{r^{\hat{d}}}{\ell + \hat{d}} \left[ \frac{i}{\omega r^{\hat{d}-1}} \left( r^{\hat{d}+1} \frac{\Lambda}{\Lambda-1} \vec{J}_w(\vec{r}) \cdot \vec{n} \right)' - r^2 \rho_\omega(\vec{r}) \right]' \\
 0 &= N_{\ell, \hat{d}} r^{2\ell+\hat{d}+1} \frac{\ell}{\ell + \hat{d}} \left( \omega^2 + \partial_r^2 + \frac{2\ell + \hat{d} + 1}{r} \partial_r \right) A_E - \rho_{L\omega}^{A_E}
 \end{aligned}$$

## Electromagnetism - reduction to gauge invariant fields 2

- Scalar (“Electric”) field

$$\begin{aligned} A_E^{L\omega} &= \left( \ell r^{\ell-1} (1 - \Lambda) \right)^{-1} \left( r^\ell (A_t^{L\omega} - i\omega A_S^{L\omega}) \right)' \\ \rho_{L\omega}^{AE} &= \int d\Omega_{\hat{d}+1} \frac{r^{\hat{d}}}{\ell + \hat{d}} \left[ \frac{i}{\omega r^{\hat{d}-1}} \left( r^{\hat{d}+1} \frac{\Lambda}{\Lambda-1} \vec{J}_w(\vec{r}) \cdot \vec{n} \right)' - r^2 \rho_\omega(\vec{r}) \right]' \\ 0 &= N_{\ell, \hat{d}} r^{2\ell+\hat{d}+1} \frac{\ell}{\ell + \hat{d}} \left( \omega^2 + \partial_r^2 + \frac{2\ell + \hat{d} + 1}{r} \partial_r \right) A_E - \rho_{L\omega}^{AE} \end{aligned}$$

- Vector (“Magnetic”) field

$$\begin{aligned} A_{MN}^{L\omega} &= \ell A_{VN}^{L\omega} / r \\ \rho_{L\omega}^{AMN} &= \frac{1}{\ell} r^{\hat{d}+2} \int \vec{J}_w(\vec{r}) \cdot \left( *(\vec{r} \wedge \vec{\nabla}) \right)_N^{XL} d\Omega_{\hat{d}+1} \\ 0 &= N_{\ell, \hat{d}} r^{2\ell+\hat{d}+1} \frac{\ell + \hat{d}}{\ell} \left( \omega^2 + \partial_r^2 + \frac{2\ell + \hat{d} + 1}{r} \partial_r \right) A_{MN} - \rho_{L\omega}^{AMN} \end{aligned}$$

## Electromagnetism - reduction to gauge invariant fields 2

More  $(2 \times L \times \omega)$  fields, but 1d ; Different wave equations

- Scalar (“Electric”) field

$$\begin{aligned} A_E^{L\omega} &= \left( \ell r^{\ell-1} (1 - \Lambda) \right)^{-1} \left( r^\ell (A_t^{L\omega} - i\omega A_S^{L\omega}) \right)' \\ \rho_{L\omega}^{A_E} &= \int d\Omega_{\hat{d}+1} \frac{r^{\hat{d}}}{\ell + \hat{d}} \left[ \frac{i}{\omega r^{\hat{d}-1}} \left( r^{\hat{d}+1} \frac{\Lambda}{\Lambda-1} \vec{J}_w(\vec{r}) \cdot \vec{n} \right)' - r^2 \rho_\omega(\vec{r}) \right]' \\ 0 &= N_{\ell, \hat{d}} r^{2\ell+\hat{d}+1} \frac{\ell}{\ell + \hat{d}} \left( \omega^2 + \partial_r^2 + \frac{2\ell + \hat{d} + 1}{r} \partial_r \right) A_E - \rho_{L\omega}^{A_E} \end{aligned}$$

- Vector (“Magnetic”) field

$$\begin{aligned} A_{MN}^{L\omega} &= \ell A_{VN}^{L\omega} / r \\ \rho_{L\omega}^{A_{MN}} &= \frac{1}{\ell} r^{\hat{d}+2} \int \vec{J}_w(\vec{r}) \cdot \left( *(\vec{r} \wedge \vec{\nabla}) \right)_N^{XL} d\Omega_{\hat{d}+1} \\ 0 &= N_{\ell, \hat{d}} r^{2\ell+\hat{d}+1} \frac{\ell + \hat{d}}{\ell} \left( \omega^2 + \partial_r^2 + \frac{2\ell + \hat{d} + 1}{r} \partial_r \right) A_{MN} - \rho_{L\omega}^{A_{MN}} \end{aligned}$$

## EM Feynman rules in general d

Directed propagator, from solution of wave equation

$$\begin{aligned}
 \text{Diagram: } & \text{A directed line from } r' \text{ to } r, \text{ with a wavy line above it.} \\
 G_{\text{ret}}^{A_E/A_M N}(r', r) = & -i \omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R_1^{E/M} \tilde{j}_\alpha(\omega r_<) \tilde{h}_\alpha^+(\omega r_>) \delta_{LL'} , \\
 R_1^E = & \frac{\ell+\hat{d}}{\ell}, \quad R_1^M = 1/R_1^M, \quad \alpha = \ell + \frac{\hat{d}}{2}, \quad M_{\ell,\hat{d}} = \frac{\pi}{2^{2\alpha+1} N_{\ell,\hat{d}} \Gamma^2(\alpha+1)}
 \end{aligned}$$

## EM Feynman rules in general d

Directed propagator, from solution of wave equation

$$\begin{aligned}
 \text{Diagram: } & \text{A directed line from } r' \text{ to } r, \text{ with a wavy line segment between them.} \\
 = G_{\text{ret}}^{A_E/A_M N}(r', r) &= -i \omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R_1^{E/M} \tilde{j}_\alpha(\omega r_<) \tilde{h}_\alpha^+(\omega r_>) \delta_{LL'} , \\
 R_1^E &= \frac{\ell+\hat{d}}{\ell}, \quad R_1^M = 1/R_1^E, \quad \alpha = \ell + \frac{\hat{d}}{2}, \quad M_{\ell,\hat{d}} = \frac{\pi}{2^{2\alpha+1} N_{\ell,\hat{d}} \Gamma^2(\alpha+1)}
 \end{aligned}$$

“Electric” & “Magnetic” source vertices

$$\begin{aligned}
 \text{Diagram: } & \text{A vertical line with a wavy line segment attached to its right end.} \\
 \equiv -Q_{L\omega}^{(E)} &= - \int dr' \tilde{j}_\alpha(\omega r') \rho_{L\omega}^{A_E}(r') \\
 &= \frac{1}{\ell+\hat{d}} \int d^D x' x'_L \left[ i\omega \tilde{j}_\alpha(\omega r') \vec{J}_\omega(\vec{x}') \cdot \vec{x}' - \frac{1}{r'^{\ell+\hat{d}-1}} \left( r'^{\ell+\hat{d}} \tilde{j}_\alpha(\omega r') \right)' \rho_\omega(\vec{x}') \right] \\
 \text{Diagram: } & \text{A vertical line with a wavy line segment attached to its left end.} \\
 \equiv -Q_{L\omega}^{(M,N)} &= - \int d^D x \tilde{j}_\alpha(\omega r) \left( *(\vec{r} \wedge \vec{J}_w(\vec{r})) \right)_{(k_\ell x_{L-1})}^N
 \end{aligned}$$

## EM in general d - results

Radiation-Reaction effective action:

$$\hat{S}_{RR} = \left\| A_E \right\| + \left\| A_M \right\| = \sum \hat{Q} G Q^*$$

## EM in general d - results

Radiation-Reaction effective action:

$$\begin{aligned}
 \hat{S}_{RR} &= \left\| \int d\omega A_E \right\| + \left\| \int d\omega A_M \right\| = \sum \hat{Q} G Q^* \\
 &= \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_{L,L'} \left[ \hat{Q}_{L\omega}^{(E)} G_{ret}^{AE}(0,0) Q_{L'\omega}^{(E)*} + \hat{Q}_{L\omega}^{(M,N)} G_{ret}^{AMN}(0,0) Q_{L'\omega}^{(M,N)*} \right] + c.c.
 \end{aligned}$$

In even d:

$$= \int dt \sum_L \frac{(-)^{\ell+\hat{d}}}{\hat{d}!!(2\ell+\hat{d})!!} \left[ \frac{\ell+\hat{d}}{\ell} \hat{Q}_L^{(E)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(E)}^L + \frac{\ell^2 \hat{d} \hat{Q}_L^{(M)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(M)}^L}{(\ell+1)(\ell+\hat{d}-1)} \right]$$

## EM in general d - results

Radiation-Reaction effective action:

$$\begin{aligned}\hat{S}_{RR} &= \left\| \int d\omega A_E \right\| + \left\| \int d\omega A_M \right\| = \sum \hat{Q} G Q^* \\ &= \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_{L,L'} \left[ \hat{Q}_{L\omega}^{(E)} G_{ret}^{A_E}(0,0) Q_{L'\omega}^{(E)*} + \hat{Q}_{L\omega}^{(M,N)} G_{ret}^{A_{MN}}(0,0) Q_{L'\omega}^{(M,N)*} \right] + c.c.\end{aligned}$$

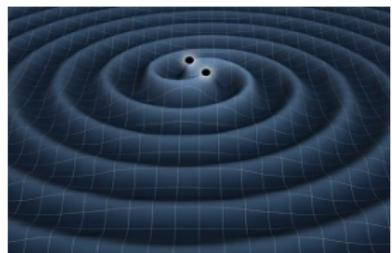
In even d:

$$= \int dt \sum_L \frac{(-)^{\ell+\frac{d}{2}}}{\hat{d}!!(2\ell+\hat{d})!!} \left[ \frac{\ell+\hat{d}}{\ell} \hat{Q}_L^{(E)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(E)}^L + \frac{\ell^2 \hat{d} \hat{Q}_L^{(M)} \cdot \partial_t^{2\ell+\hat{d}} Q_{(M)}^L}{(\ell+1)(\ell+\hat{d}-1)} \right]$$

Radiation-Reaction (Self-) Force (on a point-charge in d dimensions)

$$\vec{F}_{ALD} = \frac{\delta \hat{S}_{RR}}{\delta \vec{x}} = q^2 \frac{(-)^{\frac{d}{2}} (d-2)}{(d-1)!!(d-3)!!} \partial_t^{d-1} \vec{x} + (6 '1PN' terms) + \dots$$

## Gravitation - GW from a 2-body system

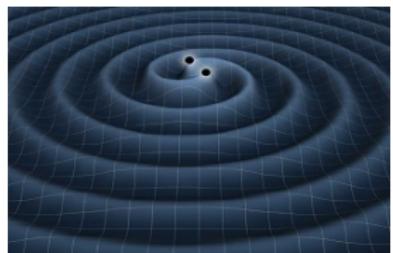


$$S = \frac{1}{16\pi G_d} \int \sqrt{-g} R d^d x - \frac{1}{2} \int h_{\mu\nu} T^{\mu\nu} d^d x,$$

$$\nabla_\mu T^{\mu\nu} = 0, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

( $c = 1$ , linearized source)

## Gravitation - GW from a 2-body system



$$S = \frac{1}{16\pi G_d} \int \sqrt{-g} R d^d x - \frac{1}{2} \int h_{\mu\nu} T^{\mu\nu} d^d x,$$

$$\nabla_\mu T^{\mu\nu} = 0, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

(  $c = 1$ , linearized source )

The same as EM fields and sources, but different

- Radiation zone:  $A_{L\omega}^{(E/M)}(r) \rightarrow h_{L\omega}^{(E/M/T)}(r)$  (gauge-invariant fields)
- System zone:  $h_{\mu\nu} \leftrightarrow (\phi, A_i, \sigma_{ij})$  (NRG fields, Kol & Smolkin '07)

these highlight spatial tensor structure, hierarchy in terms of PN

$$ds^2 = e^{2\phi} \left( dt - A_i dx^i \right)^2 - e^{-2\phi} \gamma_{ij} dx^i dx^j$$

## Gravitation - 1D reduced action & Feynman rules

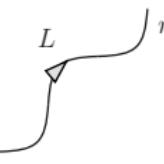
Goal - like EM:

$$S_{(E/M/T)} = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L \int dr \left[ N_{\ell,\hat{d}} \frac{r^{2\ell+\hat{d}+1}}{R_{\ell,\hat{d}}^\epsilon} h_\epsilon^{L\omega*} \left( \omega^2 + \partial_r^2 + \frac{2\ell+\hat{d}+1}{r} \partial_r \right) h_{L\omega}^\epsilon \right. \\ \left. - (h_\epsilon^{L\omega*} \mathcal{T}_{L\omega}^\epsilon + c.c.) \right]$$

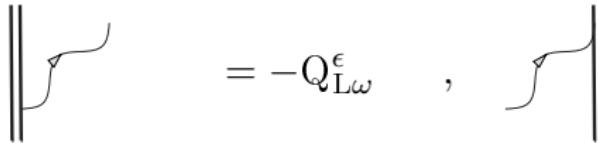
## Gravitation - 1D reduced action & Feynman rules

Goal - like EM:

$$S_{(E/M/T)} = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L \int dr \left[ N_{\ell,\hat{d}} \frac{r^{2\ell+\hat{d}+1}}{R_{\ell,\hat{d}}^\epsilon} h_\epsilon^{L\omega*} \left( \omega^2 + \partial_r^2 + \frac{2\ell+\hat{d}+1}{r} \partial_r \right) h_\epsilon^{L\omega} \right. \\ \left. - (h_\epsilon^{L\omega*} \mathcal{T}_{L\omega}^\epsilon + c.c.) \right]$$



$$= G_{\text{ret}}^\epsilon(r', r) = -i G \omega^{2\ell+\hat{d}} M_{\ell,\hat{d}} R_{\ell,\hat{d}}^\epsilon \tilde{j}_\alpha(\omega r_<) \tilde{h}_\alpha^+(\omega r_>) \delta_{LL'}$$



$$= -Q_{L\omega}^\epsilon \quad , \quad = -\hat{Q}_{L\omega}^{\epsilon*}$$

## Gravitation - special case d = 4

No tensor modes!

$$\begin{aligned} S_{(E/M)} &= \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L \int dr \left[ \frac{r^{2\ell+2}}{R^\epsilon(2\ell+1)!!} h_{(E/M)}^{L\omega*} \left( \omega^2 + \partial_r^2 + \frac{2(\ell+1)}{r} \partial_r \right) h_{L\omega}^{(E/M)} \right. \\ &\quad \left. - \left( h_{(E/M)}^{L\omega*} \mathcal{T}_{L\omega}^{(E/M)} + c.c. \right) \right], \\ R^\epsilon &:= \frac{\ell+2}{\ell-1} \left( \frac{\ell+1}{\ell} \right)^\epsilon, \quad \epsilon = 1 \text{ (E)}, \quad \epsilon = -1 \text{ (M)} \end{aligned}$$

## Gravitation - special case d = 4

No tensor modes!

$$S_{(E/M)} = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_L \int d\mathbf{r} \left[ \frac{r^{2\ell+2}}{R^\epsilon (2\ell+1)!!} h_{(E/M)}^{L\omega*} \left( \omega^2 + \partial_r^2 + \frac{2(\ell+1)}{r} \partial_r \right) h_{L\omega}^{(E/M)} \right. \\ \left. - \left( h_{(E/M)}^{L\omega*} \mathcal{T}_{L\omega}^{(E/M)} + c.c. \right) \right],$$

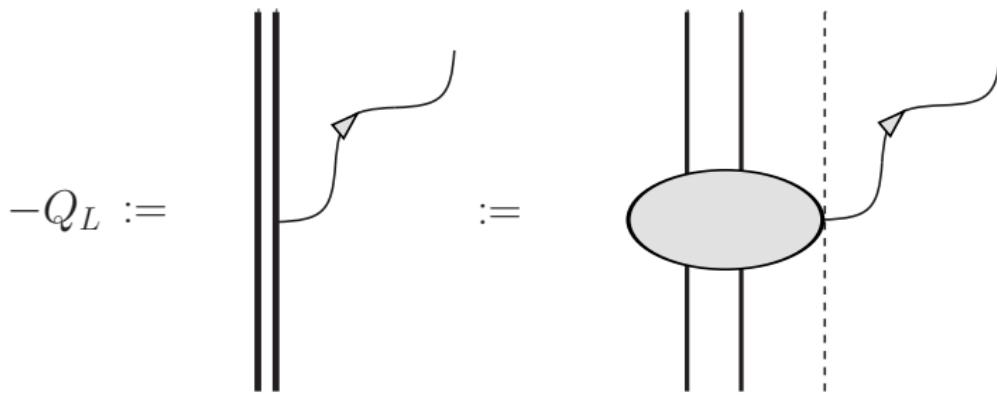
$$R^\epsilon := \frac{\ell+2}{\ell-1} \left( \frac{\ell+1}{\ell} \right)^\epsilon \quad , \quad \epsilon = 1 \text{ (E)} , \epsilon = -1 \text{ (M)}$$

$$= G_{\text{ret}}^\epsilon(r', r) = \frac{-iG\omega^{2\ell+1}}{(2\ell+1)!!} \tilde{j}_{\ell+\frac{1}{2}}(\omega r_<) \tilde{h}_{\ell+\frac{1}{2}}^+(\omega r_>) R^\epsilon$$

$$\text{Diagram} = -Q_L^\epsilon \omega \quad , \quad \text{Diagram} = -\hat{Q}_L^{\epsilon*} \omega$$

## Gravitation - Vertices via zone separation

- Work in radiation zone  $\Rightarrow$  eliminate system zone
- System zone itself includes non-linear vertices; grouped together (by PN order)
- Radiation-zone vertices defined as



## Gravitation (4d) - results

$$\begin{aligned}\hat{S}_{\text{linear}} &= \left\| h_E \right\| + \left\| h_M \right\| \\ &= \int dt \sum_{\ell} \frac{G(-)^{\ell+1} (\ell+2)}{(2\ell+1)!! (\ell-1)} \left[ \frac{(\ell+1)}{\ell} \hat{Q}_{(E)}^L \partial_t^{2\ell+1} Q_L^{(E)} \right. \\ &\quad \left. + \frac{\ell}{(\ell+1)} \hat{Q}_{(M)}^L \partial_t^{2\ell+1} Q_L^{(M)} \right]\end{aligned}$$

## Gravitation (4d) - results

$$\begin{aligned}\hat{S}_{\text{linear}} &= \left\| h_E \right\| + \left\| h_M \right\| \\ &= \int dt \sum_{\ell} \frac{G(-)^{\ell+1} (\ell+2)}{(2\ell+1)!! (\ell-1)} \left[ \frac{(\ell+1)}{\ell} \hat{Q}_{(E)}^L \partial_t^{2\ell+1} Q_L^{(E)} \right. \\ &\quad \left. + \frac{\ell}{(\ell+1)} \hat{Q}_{(M)}^L \partial_t^{2\ell+1} Q_L^{(M)} \right]\end{aligned}$$

As promised.

## Gravitation (4d) - results LO

$$\hat{S}_{\text{LO}} = -G \int dt \frac{1}{5} \hat{Q}_E^{ij} \partial_t^5 Q_E^{ij}$$

- $Q_E^{ij}$  : Mass quadrupole  $\implies$  Burke-Thorne potential & self-force

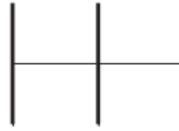
## Gravitation (4d) - results LO, NLO

$$\hat{S}_{\text{LO}} = -G \int dt \frac{1}{5} \hat{Q}_E^{ij} \partial_t^5 Q_E^{ij}$$

- $Q_E^{ij}$  : Mass quadrupole  $\Rightarrow$  Burke-Thorne potential & self-force

$$\hat{S}_{\text{LO+NLO}} = G \int dt \left[ -\frac{1}{5} \hat{Q}_E^{ij} \partial_t^5 Q_E^{ij} - \frac{4}{45} \hat{Q}_M^{ij} \partial_t^5 Q_M^{ij} + \frac{1}{189} \hat{Q}_E^{ijk} \partial_t^7 Q_E^{ijk} \right]$$

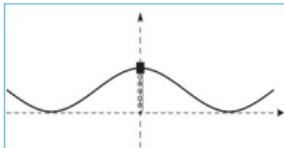
- $Q_E^{ij}$  : Mass quadrupole (+1PN corrected including first system zone nonlinear effect: gravitating potential energy,  $\sim -\frac{G m_A m_B}{r}$ )



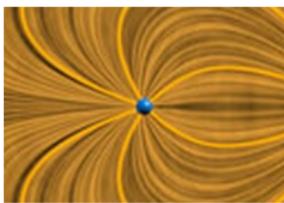
- $Q_M^{ij}$  : Current quadrupole
- $Q_E^{ijk}$  : Mass octupole

“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein

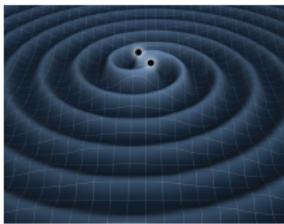
$$\hat{S} = \left\| \text{path} \right\| = \sum (\hat{Q} G Q) , \quad F = \frac{\delta \hat{S}}{\delta \hat{x}}$$



Bead + string (elastic field)



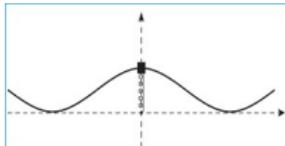
Electric charge + EM field



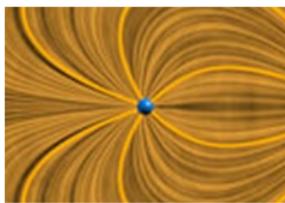
2-Body system + GR field

“Everything should be made as simple as possible,  
but not simpler” - Albert Einstein

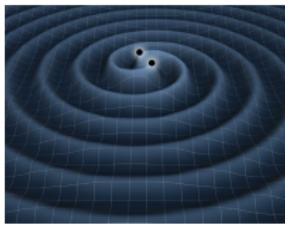
$$\hat{S} = \left\| \text{wavy path} \right\| = \sum (\hat{Q} G Q) , \quad F = \frac{\delta \hat{S}}{\delta \hat{x}}$$



$$\hat{S} = -2 Z \int \hat{y} \dot{y} dt$$



$$\hat{S} = \int dt \sum_L \frac{(-)^{\ell + \hat{d} + 1}}{\hat{d}!!(2\ell + \hat{d})!!} \left[ \frac{\ell + \hat{d}}{\ell} \hat{Q}_{(E)}^L \cdot \partial_t^{2\ell + \hat{d}} Q_{(E)}^L + \frac{\ell^2 \hat{d} \hat{Q}_{(M)}^{(M)} \cdot \partial_t^{2\ell + \hat{d}} Q_{(M)}^L}{(\ell + 1)(\ell + \hat{d} - 1)} \right]$$



$$\hat{S} = \int dt \sum_L \frac{G (-)^{\ell+1} (\ell+2)}{(2\ell+1)!!(\ell-1)} \left[ \frac{\ell+1}{\ell} \hat{Q}_{(E)}^L \partial_t^{2\ell+1} Q_{(E)}^L + \frac{\ell}{\ell+1} \hat{Q}_{(M)}^L \partial_t^{2\ell+1} Q_{(M)}^L \right]$$

## Conclusions

- Unified description of different physical systems
- Joint analytical Action formulation of radiation & reaction
- EM (ALD): New results in general dimension
- GR:
  - Economization of traditional computations
  - Nonlinear effects
  - Coming soon: leading, +1PN, some +2PN in any dimension

# Questions?

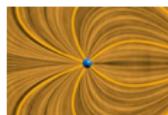


Thank you for your attention

# Image credits



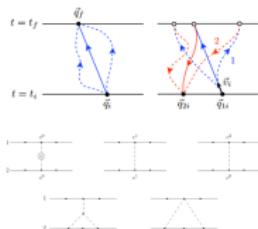
K. Thorne (Caltech) & T. Carnahan (NASA GSFC)



MIT OpenCourseWare Electrostatic Visualizations



Flickr's [Flood G](#), (creative commons license)

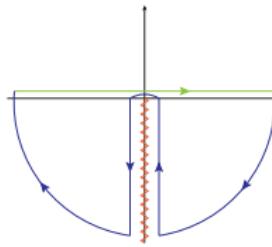


C. R. Galley, PRL 110 (2013) 17, 174301

W. D. Goldberger & I. D. Rothstein, PRD 73 (2006) 104029

## EM in non-even d

- In frequency domain - SAME.
- Transforming to time domain  $\Rightarrow$  branch-cut  $\Rightarrow$  non-locality



$$\hat{Q}(t) \partial_t^{2\ell+\hat{d}} Q(t) \rightarrow \hat{Q}(t) \left[ \left( \frac{1}{2} H(2\ell + \hat{d}) - H(\ell + \frac{\hat{d}}{2}) \right) \partial_t^{2\ell+\hat{d}} Q^L(t) - \int_{-\infty}^t \frac{dt'}{t-t'} \partial_{t'}^{2\ell+\hat{d}} Q(t') \Big|_{\text{Reg}} \right]$$

Reg.: Detweiler-Whiting decomposition, generalizes Dirac's odd propagator



## Gravitation - Full source vertices

$$\begin{aligned}
 Q_{(E)}^L &= \frac{1}{(\ell+1)(\ell+2)} \int d^3x x^L \left[ r^{-\ell} \left( r^{\ell+2} \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) \right)^{''} (T^{00} + T^{aa}) \right. \\
 &\quad \left. - \frac{4}{r^{\ell+1}} \left( r^{\ell+2} \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) \right)' \partial_t T^{0a} x^a + 2 \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) \partial_t^2 T^{ab} x^a x^b + r^2 \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) \partial_t^2 (T^{00} - T^{aa}) \right] \\
 &= \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \left( 1 + \frac{8p(\ell+p+1)}{(\ell+1)(\ell+2)} \right) \left[ \int d^3x \partial_t^{2p} T^{00} r^{2p} x^L \right]^{\text{STF}} \\
 &\quad + \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \left( 1 + \frac{4p}{(\ell+1)(\ell+2)} \right) \left[ \int d^3x \partial_t^{2p} T^{aa} r^{2p} x^L \right]^{\text{STF}} \\
 &\quad - \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \frac{4}{\ell+1} \left( 1 + \frac{2p}{\ell+2} \right) \left[ \int d^3x \partial_t^{2p+1} T^{0a} r^{2p} x^{aL} \right]^{\text{STF}} \\
 &\quad + \sum_{p=0}^{\infty} \frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!} \frac{2}{(\ell+1)(\ell+2)} \left[ \int d^3x \partial_t^{2p+2} T^{ab} r^{2p} x^{abL} \right]^{\text{STF}} \\
 Q_{(M)}^L &= \frac{1}{\ell+2} \int d^3x \left\{ r^{-\ell-1} \left( r^{\ell+2} \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) \right)' \left( 2 \vec{x} \times (\vec{T}^{0a}) \right)^{k_\ell} x^{L-1} \right. \\
 &\quad \left. - \tilde{j}_{\ell+\frac{1}{2}}(ri\partial_t) 2\epsilon^{k_\ell ba} \partial_t T^{ac} x^{bcL-1} \right\}
 \end{aligned}$$

## Origin-normalized Bessel functions

From Bessel's functions  $B_\alpha \equiv \{J_\alpha, Y_\alpha, H_\alpha^\pm\}$ ,

$$\tilde{b}_\alpha := \Gamma(\alpha + 1) 2^\alpha \frac{B_\alpha(x)}{x^\alpha}$$

They satisfy

$$\left[ \partial_x^2 + \frac{2\alpha + 1}{x} \partial_x + 1 \right] \tilde{b}_\alpha(x) = 0$$

$\tilde{j}_\alpha$  in the vicinity of the origin  $x = 0$  is given by

$$\tilde{j}_\alpha(x) = \sum_{p=0}^{\infty} \frac{(-)^p (2\alpha)!!}{(2p)!!(2p + 2\alpha)!!} x^{2p}$$

The outgoing waves at  $x \rightarrow \infty$  are  $\tilde{h}_\alpha^\pm := \tilde{j}_\alpha \pm i\tilde{y}_\alpha$ :

$$\tilde{h}_\alpha^\pm(x) \sim (\mp i)^{\alpha+1/2} \frac{2^{\alpha+1/2} \Gamma(\alpha + 1)}{\sqrt{\pi}} \frac{e^{\pm ix}}{x^{\alpha+1/2}}$$

## Singular origin-normalized Bessel functions

Around  $x = 0$  the Bessel function of the 2nd kind for  $\alpha$  non-integer is

$$\begin{aligned}\tilde{y}_\alpha(x) &= \frac{\Gamma(\alpha + 1)2^\alpha}{x^\alpha} \frac{\cos(\alpha\pi)J_\alpha(x) - J_{-\alpha}(x)}{\sin(\alpha\pi)} \\ &= \frac{\Gamma(\alpha + 1)2^\alpha}{\sin(\alpha\pi)} \sum_{p=0}^{\infty} \frac{(-)^p}{(2p)!!} \frac{x^{2p}}{2^p} \left[ \frac{\cos(\alpha\pi)}{2^\alpha \Gamma(p + \alpha + 1)} - \frac{2^\alpha x^{-2\alpha}}{\Gamma(p - \alpha + 1)} \right]\end{aligned}$$

while for integer  $\alpha = n$ ,

$$\begin{aligned}\tilde{y}_n(x) &= -\frac{2(2n)!!}{\pi} \sum_{m=1}^n \frac{(2m - 2)!!}{(2n - 2m)!!} x^{-2m} + \frac{2}{\pi} \ln\left(\frac{x}{2}\right) \tilde{j}_n(x) \\ &\quad - \frac{(2n)!!}{\pi} \sum_{k=0}^{\infty} [\psi(k + 1) + \psi(n + k + 1)] \frac{(-)^k}{(2k)!!(2n + 2k)!!} x^{2k}\end{aligned}$$

where  $\psi(N + 1) = H(N) - \gamma$  is the digamma function, defined using the Harmonic numbers  $H(N)$  and the Euler-Mascheroni constant  $\gamma$ .

## Effective Field Theory description of GR

For example, Einstein-Infeld-Hoffmann Lagrangian

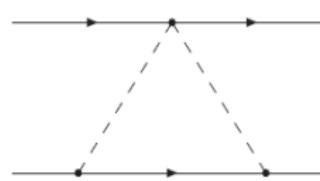
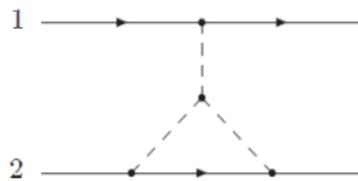
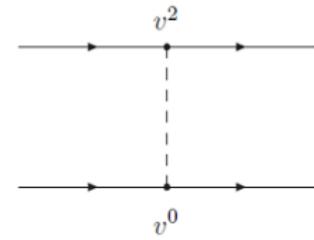
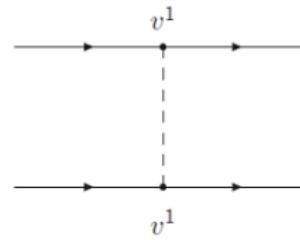
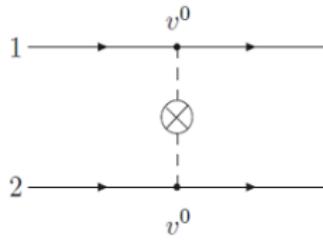
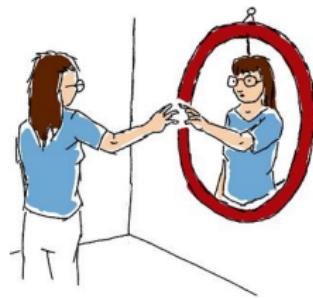


Figure from W. D. Goldberger and I. D Rothstein, PRD 73 (2006) 104029

## Hatted variables - interpretation

Doubling the fields effectively “mirrors” them.



Hatted fields "pull" from the mirror's surface, absorbing radiation from the original fields.

$$0 = \frac{\delta \hat{S}}{\delta \phi(x)} = \int dx' \frac{\delta \text{EOM}_\phi(x')}{\delta \phi(x)} \hat{\phi}(x') - \hat{\rho}(x)$$

- $\hat{\phi}$  satisfies an equation for linearized deviations from the background  $\phi$  with a source  $\hat{\rho}$  and **reversed** propagation.
- In the absence of its source  $\hat{\phi}$  vanishes, namely

$$\hat{\rho} = 0 \implies \hat{\phi} = 0$$

- $\phi, \rho$  are physical observables, unlike  $\hat{\phi}, \hat{\rho}$ .

## Field doubling

- Problem: Action formalism not compatible with non-conservative effects.
- Solution: Field Doubling (QFT: in-in/CTP, Schwinger '61; Classical version: Galley 2012, 2013).

$$\hat{S} := S[q_1] - S[q_2]$$

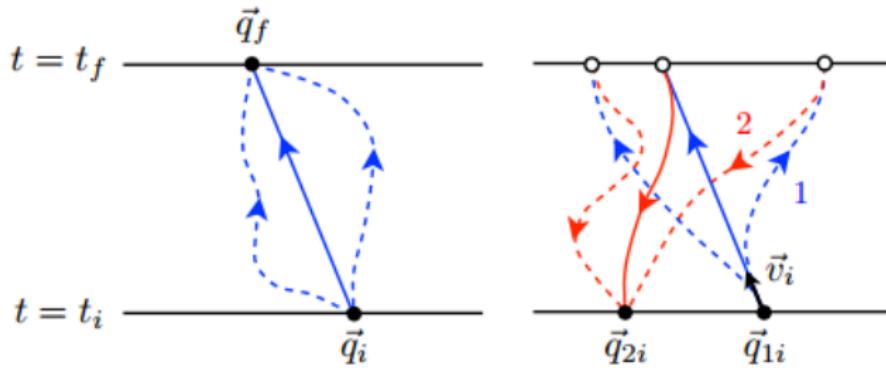


Figure from C. R. Galley, PRL 110 (2013) 174301

## Field doubling

- Keldysh representation:

$$q_1 := q + \frac{1}{2} \hat{q}$$

$$q_2 := q - \frac{1}{2} \hat{q}$$

- $\hat{q}$  is deviation from physical trajectory. Expand:

$$\hat{S} := S[q_1] - S[q_2] = \frac{\delta S}{\delta q} \hat{q} + \frac{1}{6} \frac{\delta^3 S}{\delta q^3} \hat{q}^3 + \dots$$

- In classical setting (“physical condition”  $\Leftrightarrow q_1 = q_2$ ) only terms linear in  $\hat{q}$  survive.

## Field doubling

- In our problem:

$$S[\phi, \rho] \rightarrow \hat{S}[\phi, \hat{\phi}; \rho, \hat{\rho}] := \int dx \left[ \frac{\delta S}{\delta \phi(x)} \hat{\phi}(x) + \frac{\delta S}{\delta \rho(x)} \hat{\rho}(x) \right]$$

- EOM:

$$\frac{\delta \hat{S}}{\delta \hat{x}} = 0$$

- Propagators are directed (retarded/odd).

## GR - NLO details

$$\hat{S} = G \int dt \left[ -\frac{1}{5} \hat{Q}_E^{ij} \partial_t^5 Q_E^{ij} - \frac{4}{45} \hat{Q}_M^{ij} \partial_t^5 Q_M^{ij} + \frac{1}{189} \hat{Q}_E^{ijk} \partial_t^7 Q_E^{ijk} \right]$$

where

$$Q_E^{ij} = \sum_{A=1}^2 m_A \left[ 1 + \frac{3}{2} v^2 - \frac{\mathbf{m}_B}{r} - \frac{4}{3} \partial_t (\vec{v} \cdot \vec{x}) + \frac{11}{42} \partial_t^2 x^2 \right]_A \left( x^i x^j - \frac{1}{3} \delta^{ij} x^2 \right)_A$$

$$Q_M^{ij} = \sum_{A=1}^2 \left[ m (\vec{x} \times \vec{v})^i x^j \right]_A^{TF}$$

$$Q_E^{ijk} = \sum_{A=1}^2 (m x^i x^j x^k)_A^{TF}$$